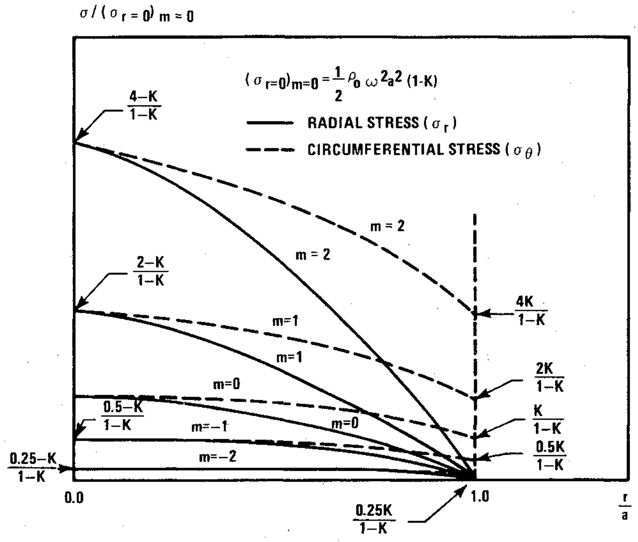

Fig. 2 Stress distrubition in rotating orthotropic disks with various densities.

When m is set equal to zero, Eq. (7) becomes

$$\sigma_r = \tfrac{1}{2}\rho_0\omega^2 a^2 (1-K)(a^2-r^2) \tag{8a}$$

$$\sigma_\theta = \tfrac{1}{2}\rho_0\omega^2 a^2 \left[K(3r^2-a^2)+(a^2-r^2) \right] \tag{8b}$$

which is the solution of a rotating orthotropic disk with constant density.[7,8]

Substituting the stress components in Eq. (7) into stress-strain relationships and integrating the results, the displacements along the principal material direction x and y are

$$U = \frac{a_{11}\rho\omega^2 x}{2}\left\{K\left(\frac{x^2}{3}+3y^2-a^2\right)-\left(\frac{x^2}{3}+y^2\right)\right\}$$

$$+2^{m-1}a_{11}\rho_0\omega^2 a^2 x + \frac{a_{12}\rho\omega^2 x}{2}\left\{K(x^2+y^2-a^2)\right.$$

$$\left.-\left(\frac{x^2}{3}+y^2\right)\right\}+2^{m-1}a_{12}\rho_0\omega^2 a^2 x \tag{9a}$$

$$V = \frac{a_{21}\rho\omega^2 y}{2}\left\{K(x^2+y^2-a^2)-\left(x^2+\frac{y^2}{3}\right)\right\}$$

$$+2^{m-1}a_{21}\rho_0\omega^2 a^2 y + \frac{a_{22}\rho\omega^2 y}{2}$$

$$\left\{K\left(3x^2+\frac{y^2}{3}-a^2\right)-\left(x^2+\frac{y^2}{3}\right)\right\}+2^{m-1}a_{22}\rho_0\omega^2 a^2 y \tag{9b}$$

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Convergence Control in Differential Dynamic Programing Applied to Air-to-Air Combat

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One particular difficulty when solving differential game problems numerically is convergence control, i.e., the measures taken between iterations to ensure the stability of the approximation and to speed up the convergence. Most often when solving practical optimization problems one of the following methods will be used: neighboring extremal methods, gradient method, quasilinearization method, or, as in the present case, differential dynamic programing (DDP).

This Note presents a new convergence control technique for DDP, which is applicable to the computation of optimal trajectories for aircraft. When evaluating such trajectories, the author first tried some algorithms of neighboring extremals. The usual difficulties due to the missing boundary values when solving TPBVP's were encountered. In particular, the convergence domain of the initial values is too small. On the other hand DDP very rapidly resulted in an acceptable solution. In DDP the initial value problem corresponds to the situation when a nominal trajectory is far away from the optimal. In this case the established way is to use the step-size-method of Jacobson-Mayne.[1] A new method—an alternative to the step-size-method—developed by the author has several advantages, particularly considering differential game problems. For a test, the new method also was applied to an example from Ref. 1. (pp. 35-38)—the Rayleigh equation. It provided good convergence.

Concerning optimal trajectories for aircraft the state vector turns out to have a dimension larger than five. This gives a strong increase in the number of equations when using a second-order expansion. In this type of problem the exact optimal trajectory need not be reached because, in practice, it cannot be implemented. Therefore, in most cases, a first-order expansion has been deemed satisfactory when using DDP with the new method, although it would work for the second-order.

The purpose of this paper is to demonstrate: 1) A method which could be an alternative to the step-size method.[1] 2) An extension of the application of DDP to differential game, air-to-air combat problems. This is done by comparison to Ref. 5 in which the gradient method and the neighboring method have been applied.

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Summary Description of Ordinary DDP

Consider a dynamic system described by

$$\dot{x}=f(x,u;t) \quad x(t_0)=x_0 \quad t\,\epsilon[t_0,T] \tag{1}$$

where x is an n-dimensional state vector and u is an r-dimensional control vector. The mixed state-control variable constraints are[2]

$$g(x,u;t)\leq 0 \tag{2}$$

The object is to minimize the cost functional

$$V(x_0;t_0)=\int_{t_0}^{T} L(x,u;t)\ \mathrm{d}t+F[x(T);T] \tag{3}$$

The optimal cost $V(x;t)$ satisfies Bellman's partial differential equation

$$(\partial/\partial t)\,V(x;t)+\underset{u}{\mathrm{Min}}\ H(x,u,V_x;t)=0 \tag{4}$$

where

$$H=L(x,u;t)+V_x^T(x;t)f(x,u;t) \tag{5}$$

T in $V_x^T(x;t)$ stands for transpose, and

$$V_x=\mathrm{col}[(\partial V/\partial x_1),\ldots,(\partial V/\partial x_n)]$$

For the reasons mentioned the first-order expansion will be considered. When Eq. (4) is expanded in $x(t)$ to the first-order, the following equations can be derived:

$$-\dot{a}=H(\bar{x},u^*,V_x;t)-H(\bar{x},\bar{u},V_x;t) \qquad a(T)=0 \tag{6}$$

$$-\dot{V}_x=H_x(\bar{x},u^*,V_x;t)+[\mu^T g_x(\bar{x},u^*;t)]^T \tag{7a}$$

$$V_x[(\bar{x}(T);T)]=F_x[\bar{x}(T);T] \tag{7b}$$

Where $\bar{u}(t)$ is a nominal control causing a nominal trajectory $\bar{x}(t)$ by Eq. (1); $u^*(t)$ minimizes $H(\bar{x},u,V_x;t)$ $\forall t\epsilon[t_0,T]$ subject to Eq. (2); $a(\tau)$ is defined as the difference between the optimal cost, obtained by using $u^*(t)$ in the interval $[\tau,T]$, and the nominal cost, obtained by using $\bar{u}(t)$. This holds under the assumption that $\Delta x=x^*(t)-\bar{x}(t)$ is not too large in the interval $[\tau,T]$ where $x^*(t)$ is the optimal trajectory obtained by Eq. (1) when using $u^*(t)$; μ is a multiplier function.[2]

Computational Procedure

1) Use a nominal control $\bar{u}(t)$ in Eq. (1) forward in time, obtaining a nominal trajectory $\bar{x}(t)$ which is stored together with $\bar{u}(t)$ and the corresponding computed cost, Eq. (3).

2) Integrate Eqs. (6-7) backward from T to t_0 while minimizing $H(\bar{x},u,V_x;t)$ with respect to $u(t)$, subject to Eq. (2), to obtain $u^*(t)$. Store $u^*(t)$ and $a(t)$.

3) Apply the new control $u^*(t)$ to Eqs. (1) and (3) for $t\epsilon[t_0,T]$. If the difference ΔV between the new and the nominal cost of step 1 is in the order of $a(t_0)$, the nominal control in step 1 can be exchanged for $u^*(t)$. Steps 1-3 are repeated until $a(t_0)$ is less than a specific number.

4) If ΔV deviates too much from $a(t_0)$, then use the so-called step-size adjustment method or the new convergence control method.

Step-Size Method

The reason why ΔV deviates too much from $a(t_0)$ is that $\Delta x(t)$ is too large and the series expansion of Eq. (4) is not valid. In general,

$$\Delta\dot{x}=\dot{x}^*-\dot{\bar{x}}\simeq f_x(\bar{x},\bar{u};t)\,\Delta x+f_u(\bar{x},\bar{u};t)\,\Delta u \tag{8}$$

Let us introduce $t_1\epsilon[t_0,T]$

$$\text{then} \qquad \Delta x(t_1)=0$$

$$\text{if} \quad \Delta u(t_2)=u^*(t_2)-\bar{u}(t_2)=0\ \forall\ t_2\epsilon[t_0,t_1]$$

By choosing t_1 near T, then Δx in the interval $[t_1,T]$ will be small enough, even though Δu is large. Thus when ΔV deviates from $a(t_0)$, choose

$$\bar{u}^{i+1}(t)=\bar{u}^i(t) \quad \forall t\epsilon[t_0,t_1]$$

and

$$\bar{u}^{i+1}(t)=u^{*i}(t) \quad \forall t\epsilon[t_1,T]$$

where i is the iteration index. The time t_1 is chosen by trial. For example, start with $t_1=t_0+(T-t_0)/2$, then choose $t_1=t_0+{}^3\!/\!_4\,(T-t_0)$, etc., until $\Delta V\simeq a(t_1)$.

New Convergence Control Method

The size of $\Delta x(t)$ can be restrained if $\Delta u(t)$ is not too large in the interval $[t_0,T]$. Equation (8) gives

$$|\Delta\dot{x}(t)|\leq K_x|\Delta x(t)|+K_u|\Delta u(t)| \quad \Delta x(t_0)=0 \tag{9}$$

where K_x and K_u are positive, bounded constants. Then

$$|\Delta x(t)|\leq\int_{t_0}^{t}\exp^{K_x(t-\tau)}K_u|\Delta u(\tau)|\,\mathrm{d}\tau \tag{10}$$

$\Delta u(\tau)$ will be kept small by an extra penalty term added to $L(x,u;t)$ in Eq. (3)

$$\hat{V}(x_0;t_0)=\int_{t_0}^{T}[L(\hat{x},\hat{u};t)+\Delta u^T(t)C\Delta u(t)]\,\mathrm{d}t+F(\hat{x}(T);T) \tag{11}$$

where $\Delta u(t)=\hat{u}^{*i}(t)-\hat{u}^{*i-1}(t)$; C is a diagonal matrix $\{c_{kk}\}$ and T in Δu^T stands for transpose.

It can be proved that[7]

$$\lim_{i\to\infty}\hat{V}=V \tag{12}$$

for $c_{kk}\geq 0$

Main Features of New Method

1) When u is a vector containing at least one sensitive component that may cause wild trajectories, this component can be given a stronger penalty by the corresponding component in C.

2) When stability has been reached it is possible to increase the convergence rate by decreasing C carefully from iteration to iteration.[7]

3) DDP requires that $H_{uu}(\bar{x},u^*;t)>0$, which is not always valid. In comparison, $H_{uu}(\bar{x},\hat{u}^*,C;t)$ always can be made positive by an increase of C.

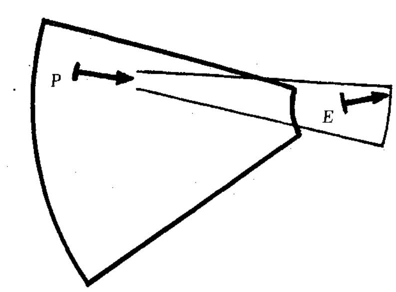

Fig. 1 Firing envelope.

4) Usually the step-size method will cause overshoot in $u^*(t)$ when t is large. However, the new method gives a more constrained $\hat{u}^*(t)$ for large t, since the large corrections in $x(t)$ that may be necessary have already been taken care of by moderate changes in $\hat{u}^*(t)$ for small t [see Eq. (10)].

5) Singular control problems will become nonsingular for some $c_{kk}>0$ (compare Refs. 3 or 6).

6) In the second-order version of DDP, when u does not influence the Riccati equation [Ref. 1, Eq. (2.2.20) p. 21], it may be necessary to start with the step-size method for a few iterations concurrently with the new method.

DDP Applied to Differential Games

In a zero-sum differential game between two players having perfect information, pursuer (P) and evader (E), [Eqs. (1) and (3)] will now be

$$\dot{x}=f(x,u,w;t) \quad x(t_0)=x_0 \quad t\epsilon[t_0,T] \tag{13}$$

$$V(x_0;t_0)=\int_{t_0}^{T} L(x,u,w;t)\,\mathrm{d}t+F(x(T);T) \tag{14}$$

where u is the control vector of P and w the control vector of E. Equation (14) is formulated so as to reflect P's trying to decrease V against E's effort to increase V.

Suppose there exist controls $u^*(t)$ and $w^*(t)$ such that

$$V^{u^*,w} \le V^{u^*,w^*} \le V^{u,w^*} \tag{15}$$

then the pair $u^*(t)$ and $w^*(t)$ is called a saddle point.

It can be shown that under certain circumstances,[4] Eq. (4) now will become

$$(\partial/\partial t)\,V(x;t)+\operatorname*{Max}_{w}\ \operatorname*{Min}_{u}\ H(x,u,w,V_x;t)=0 \tag{16}$$

and

$$\operatorname*{Max}_{w}\ \operatorname*{Min}_{u}\ H=\operatorname*{Min}_{u}\ \operatorname*{Max}_{w}\ H \Rightarrow u^*(t) \text{ and } w^*(t) \tag{17}$$

Realistic Air-to-Air Combat

Consider two aircraft, and write Eq. (13) in relative coordinates

$$\dot{x}_1=x_2\dot{\chi}_P\cos\gamma_P+v_E\cos\gamma_E\sin x_4-x_3\dot{\chi}_P\sin\gamma_P \tag{18a}$$

$$\dot{x}_2=-x_1\dot{\chi}_P\cos\gamma_P-v_p+v_E(\sin\gamma_P\sin\gamma_E+\cos\gamma_P\cos\gamma_E\cos x_4)+x_3\dot{\gamma}_P \tag{18b}$$

$$\dot{x}_3=-v_E(\sin\gamma_P\cos\gamma_E\cos x_4-\cos\gamma_P\sin\gamma_E)-x_2\dot{\gamma}_P+x_1\dot{\chi}_P\sin\gamma_P \tag{18c}$$

$$\dot{x}_4=\dot{\chi}_P-\dot{\chi}_E \tag{18d}$$

$$\dot{\gamma}_P=\ldots. \tag{18e}$$

$$\dot{\gamma}_E=\ldots. \tag{18f}$$

$$\dot{v}_P=\ldots. \tag{18g}$$

$$\dot{v}_E=\ldots. \tag{18h}$$

The equations of motion for an aircraft can generally be written

$$\dot{\chi}=(n\,g+T^M\pi\sin\alpha/m)\sin\mu/v\cos\gamma \tag{19a}$$

$$\dot{\gamma}=(n\,g+T^M\pi\sin\alpha/m)\cos\mu/v-g\cos\gamma/v \tag{19b}$$

$$\dot{v}=T^M\pi\cos\alpha/m-D_ov^2-D_in^2/v^2-g\sin\gamma \tag{19c}$$

where x_1,x_2,x_3 are the relative positions between the two vehicles; x_2 is orientated in P's flight direction, x_1 is parallel with the horizontal plane and perpendicular to x_2, and x_3 is perpendicular to x_1 and x_2; x_4 is the relative course; γ,χ,μ, and α are the climb, course, bank angle, and angle of attack, respectively; v is the speed; g is the acceleration due to gravity; m is the mass; T^M is the maximum thrust; D_0 and D_i are aerodynamic drag coefficients; $\pi\epsilon[0,1]$ is the throttle setting; n is the load factor, which is limited by $g(v,n)\le 0$ (see Ref. 5, p. 7, Fig. 2.2). The condition $g(v,n)\le 0$ is easily implemented in the DDP procedure.[2] The control variables are π,n, and μ.

The following special case of Eq. (14) has been studied

$$V(x_0;0)=\int_0^T[q_1(\pi_P^2+c_1\Delta\pi_P^2)+q_2(n_P^2+c_2\Delta n_P^2)-q_3(\pi_E^2+c_3\Delta\pi_E^2)-q_4(n_E^2+c_4\Delta n_E^2)]\,dt+F[x(T);T] \tag{20}$$

The weight factors $q_1,\ldots,q_4>0$. Of these q_1 and q_3 will penalize the fuel consumption, whereas q_2 and q_4 weight the ability of the pilots to stand a load factor for a duration of time; Δ stands for the difference in successive iterations of the control variables [see Eq. (11)]; $c_1,\ldots,c_4$ are the convergence factors; $F[x(T);T]$ describes a firing envelope, e.g., according to Fig. 1.

This problem has been programed on the CDC 6600 with a first-order DDP and the new convergence control method. In the program T^M,D_0 and D_i were held constant. However, it would probably not cause any problem to make these parameters functions of height, speed, and angle of attack.

A test was made to obtain a comparison with Ref. 5. The object of the particular example was to obtain a saddle point between a missile (P) and an aircraft (E). The firing envelope was

$$F[x(T);T]=(x_1^2+x_2^2+x_3^2)^{1/2}$$

Leatham-Lynch[5] solved this example with two methods on the CDC 6600 computer: 1) the gradient method, taking 576 cps and 2) the neighboring method typically using 100 cps. With the present DDP approach the solution was reached in 25 cps. The trajectories obtained are equal to those of Ref. 5 as far as can be judged from the plots. Reference 5 features a more complete model than Eqs. (19), e.g., by including the equation $\dot{m}=-b$. The authors also specify T^M,D_0, and D_i as functions of height, v, and α, but it is not quite clear that these variables were computed as such.

Conclusion

The new convergence control method has turned out to be a good alternative to the step-size method, which is the original method of Ref. 1. An example from Ref. 1 was tested with the new method by which faster convergence was obtained. DDP with the new method also proved efficient when applied to a problem well known for its difficulty. Experiments trying to decrease C to improve the convergence rate yielded good results. These topics as well as the range of the method are still under research.

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Application of Elasto-Plastic Finite Element Analysis to the Contact Problems of Solids

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ALTHOUGH numerous closed form solutions for contact problems of solids are available, they are limited to simple geometries and loading conditions. Advances in the finite element method have made possible almost revolutionary changes in analyses of this kind.[1-4] The concept of differential displacements was used widely in these analyses and was applied to problems of elastic-rigid body contact. A more sophisticated analysis was developed recently by Otte[5] to handle contact problems of two elastic bodies. His method was based on the concept of normal force and displacement compatibility at the contact surfaces and allows the inclusion of interface friction effects. His work has some disadvantages; it involves rather tedious formulations, is limited to elastic systems, and was not derived for curved boundaries.

The present method is based on a simple concept which involves the introduction of a very thin layer of elasto-plastic interface elements between the two contacting solids. The mechanical properties of these interface elements are selected in such a way that they represent a noncompressible fluid (ideally plastic) when transmitting compressive loadings and a gas-like material upon partial or complete separation of the contacted surfaces. No additional formulation or changes are required when using established elasto-plastic finite element analysis techniques. This approach was used successfully in simulating complicated contact-separation problems of a nuclear reactor fuel element.[6]

A case of two concentric cylinders is considered to demonstrate the application of this technique for contact problems. The inner cylinder is subjected to a contant pressure and a time-varying normal surface traction, $P_o(t)$ is applied to the outside surface of the system as shown in Fig. 1b. The two cylinders are in contact initially and the contact pressure reduces as $P_o(t)$ increases and the two cylinders will eventually separate when $P_o(t)$ reaches a certain level. The choice of this case study is partially motivated by existing closed form solutions available for checking the finite element results.

For plane strain conditions, the stress and displacement components in a hollow cylinder subject to both internal pressure and external traction P_o as shown in Fig. 1a, can be calculated from the formula[7]

$$\sigma_{rr} = \frac{a^2P_i - b^2P_o}{b^2-a^2} - \frac{(P_i-P_o)a^2b^2}{(b^2-a^2)r^2} \quad \text{(1a)}$$

$$\sigma_{\theta\theta} = \frac{a^2P_i - b^2P_o}{b^2-a^2} + \frac{(P_i-P_o)a^2b^2}{b^2-a^2}\frac{1}{r^2} \quad \text{(1b)}$$

$$u_r = \frac{1-\nu}{E}\frac{a^2P_i - b^2P_o}{b^2-a^2}r + \frac{1+\nu}{E}\frac{a^2b^2(P_i-P_o)}{b^2-a^2}\frac{1}{r} \quad \text{(1c)}$$

The contact pressure at the interface of this two cylinder system shown in Fig. 1b can be derived from the compatibility condition of radial displacement and produces

$$P_c = \frac{\dfrac{2a^2}{E_1(b^2-a^2)}P_i - \dfrac{2c^2}{E_2(c^2-b^2)}P_o}{\dfrac{1}{E_1}\left(\dfrac{b^2+a^2}{b^2-a^2}-\nu_1\right)+\dfrac{1}{E_2}\left(\dfrac{c^2+b^2}{c^2-b^2}+\nu_2\right)} \quad (2)$$

Where E is the modulus of elasticity, ν is the Poisson's ratio while the subscripts 1 and 2 designate the material properties of inner and outer cylinder, respectively.

The dimensions of the cylinders used for the numerical illustration are given in Fig. 1c. The pressure loadings are assumed to be $P_i = 74.14$ psi and $(dP_o/dt)(t) = 4$ psi/sec. The finite-element idealization for the fixed ends (plane strain) case is shown in Fig. 1c. The shaded element (no. 9) is the interface element which functions as previously described. Triangular elements were used near the interface for the better radial resolution of stresses in this region before separation takes place. A total of three stacks of elements were used in the longitudinal direction for examining the sliding of the interface as shown in Fig. 2. The two cylinders may "slide" in this simple case when contact pressure disappears. At this instant the finite element code would change material properties of the interface element as indicated in Table 1. Figure 2 shows the realtive contraction of the contact surfaces at various distances away from the fixed end. It may be noticed that relative sliding of the surfaces increases with distance

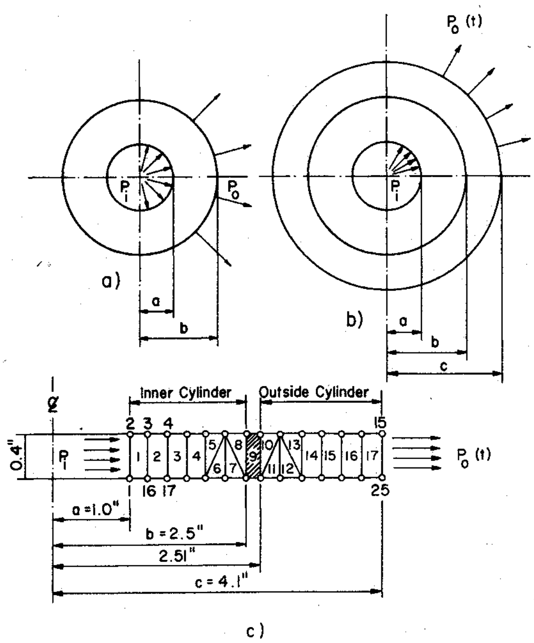

Fig. 1 Compound cylinder system and F. E. idealization.

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